[10-03-15-NAM11]

9. *ABCDEF* is a regular hexagon. Express in terms of a single vector the sum of the vectors

(a) $\overrightarrow{AB}$, $\overrightarrow{AE}$, (b) $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AE}$, $\overrightarrow{AF}$,

(c) $\overrightarrow{AB}$, $\overrightarrow{AF}$, (d) $4\overrightarrow{AB}$, $2\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{AE}$, $5\overrightarrow{AF}$. *(C)*

10. *ABCDEF* is a regular hexagon. Given that $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{BC} = \mathbf{q}$, express the following in terms of one, or both, **p** and **q**.

(a) $\overrightarrow{AD} + \overrightarrow{BE}$ (b) $\overrightarrow{BD} + \overrightarrow{CE}$

23.2 VECTORS EXPRESSED IN TERMS OF TWO NON-PARALLEL VECTORS

In Example 4, **p** and **q** are non-parallel vectors and

$\overrightarrow{FD} = \mathbf{p} + \mathbf{q}$,

$\overrightarrow{CD} = \mathbf{q} - \mathbf{p} = -\mathbf{p} + \mathbf{q}$,

$\overrightarrow{BE} = 2(\mathbf{q} - \mathbf{p}) = -2\mathbf{p} + 2\mathbf{q}$.

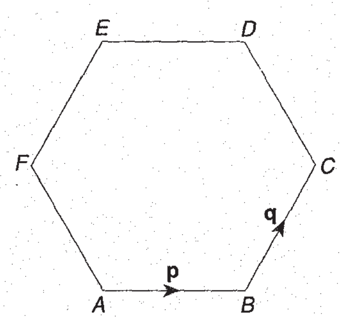


Fig. 23.16

Each of these vectors is expressed in terms of the non-parallel vectors **p** and **q** and is of the form $m\mathbf{p} + n\mathbf{q}$, where m and n are scalars.

In general:

> If **a** and **b** are two non-zero and non-parallel vectors, any vector $\overrightarrow{OP}$ in the plane containing **a** and **b** can be expressed in terms of **a** and **b**. That is,
>
> $\overrightarrow{OP} = m\mathbf{a} + n\mathbf{b}$, where m and n are constants.

This result may be explained as follows:

For any vector $\overrightarrow{OP}$, there is a parallelogram $OA'PB'$ such that

$$\overrightarrow{OA'} = m\mathbf{a}, \quad \overrightarrow{OB'} = n\mathbf{b}$$

and

$$\overrightarrow{OP} = \overrightarrow{OA'} + \overrightarrow{OB'} \quad \textit{(parallelogram rule)}$$
$$= m\mathbf{a} + n\mathbf{b}$$

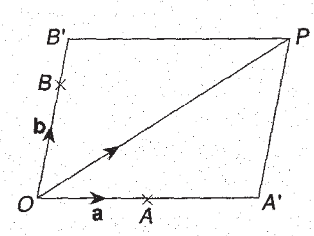


Fig. 23.17

An important nature of vectors is that two non-parallel vectors cannot be equal. This enables us to have the following results:

If non-zero vectors **a** and **b** are non-parallel, then

(a) $\lambda \mathbf{a} = \mu \mathbf{a} \Rightarrow \lambda \mathbf{a} = \mathbf{0}$ and $\mu \mathbf{a} = \mathbf{0}$
$\Rightarrow \lambda = 0$ and $\mu = 0$,

(b) $p\mathbf{a} + q\mathbf{b} = r\mathbf{a} + s\mathbf{b} \Rightarrow (p - r)\mathbf{a} = (s - q)\mathbf{b}$
$\Rightarrow (p - r) = 0$ and $(s - q) = 0$,

which gives

$$p\mathbf{a} + q\mathbf{b} = r\mathbf{a} + s\mathbf{b} \Rightarrow p = r \text{ and } q = s.$$

Example 6 Given that vectors **a** and **b** are non-parallel and non-zero, find the values of t and s if

$$\mathbf{a} + t(\mathbf{b} + 2\mathbf{a}) = 2\mathbf{a} + \mathbf{b} + s(\mathbf{a} - \mathbf{b}).$$

Solution:

$\mathbf{a} + t(\mathbf{b} + 2\mathbf{a}) = 2\mathbf{a} + \mathbf{b} + s(\mathbf{a} - \mathbf{b})$
$(1 + 2t)\mathbf{a} + t\mathbf{b} = (2 + s)\mathbf{a} + (1 - s)\mathbf{b}$
$\Rightarrow \quad 1 + 2t = 2 + s$
$2t = 1 + s$ .... (1)
and $t = 1 - s$ .... (2)
Adding (1) and (2),

$$3t = 2 \Rightarrow t = \frac{2}{3} \text{ and so } s = \frac{1}{3}$$

Example 7 In Fig. 23.18, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OP} = 2\mathbf{a} + 3\mathbf{b}$. Given that $\overrightarrow{OM} = \lambda\overrightarrow{OP}$ and $\overrightarrow{AM} = \mu\overrightarrow{AB}$, express $\overrightarrow{OM}$

(a) in terms of λ, **a** and **b**,

(b) in terms of μ, **a** and **b**.

Hence find the values of λ and μ and express $\overrightarrow{OM}$ in terms of **a** and **b**.

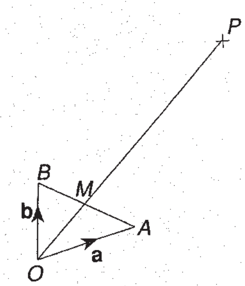


Fig. 23.18

Solution:

(a) $\overrightarrow{OM} = \lambda\overrightarrow{OP}$
$= \lambda(2\mathbf{a} + 3\mathbf{b})$ .... (1)

(b) $\overrightarrow{AM} = \mu\overrightarrow{AB}$
$= \mu(\overrightarrow{OB} - \overrightarrow{OA})$
$= \mu(\mathbf{b} - \mathbf{a})$

By the triangle law,

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$$
$$= \mathbf{a} + \mu(\mathbf{b} - \mathbf{a})$$
$$= \underline{\underline{(1 - \mu)\mathbf{a} + \mu\mathbf{b}}} \quad \text{.............................} \quad (2)$$

From (1) and (2),

$$2\lambda\mathbf{a} + 3\lambda\mathbf{b} = (1 - \mu)\mathbf{a} + \mu\mathbf{b}$$
$$\Rightarrow \quad 2\lambda = 1 - \mu \text{ and } 3\lambda = \mu$$
$$\Rightarrow \quad \lambda = \underline{\underline{\frac{1}{5}}}, \ \mu = \underline{\underline{\frac{3}{5}}}$$

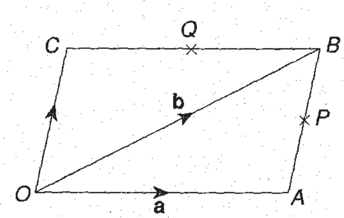
and $\overrightarrow{OM} = \frac{1}{5}(2\mathbf{a} + 3\mathbf{b})$

$$= \underline{\underline{\frac{2}{5}\mathbf{a} + \frac{3}{5}\mathbf{b}}}$$

Exercise 23.2

(Answers on p. 590)

1. $OABC$ is a parallelogram with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. P and Q are the midpoints of AB and BC respectively. Find $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{AQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.



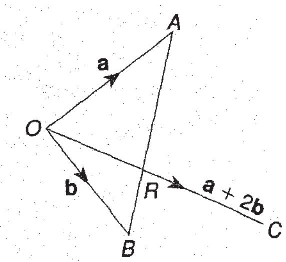
2. $ABCDEF$ is a regular hexagon with centre O. Suppose $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{AF} = \mathbf{q}$, express $\overrightarrow{AO}$, $\overrightarrow{AC}$, $\overrightarrow{AE}$ and $\overrightarrow{CE}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

3. Given that the non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are non-parallel and that
$$3\mathbf{a} + t(2\mathbf{a} - 3\mathbf{b}) = \mathbf{a} + \mathbf{b} + s(\mathbf{a} + 2\mathbf{b}),$$
find the value of t and of s.

4. Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are non-parallel. If $2\mathbf{a} + t(\mathbf{a} - \mathbf{b})$ and $2\mathbf{b} + \mathbf{a} + t\mathbf{b}$ are parallel, find the values of t.

5. $OABC$ is a square with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. M is the midpoint of BC and AM intersects OC produced at P. Given that $\overrightarrow{AP} = k\overrightarrow{AM}$ and $\overrightarrow{OP} = n\overrightarrow{OC}$, express $\overrightarrow{OP}$
   (a) in terms of k, $\mathbf{a}$ and $\mathbf{c}$,
   (b) in terms of n, $\mathbf{a}$ and $\mathbf{c}$.
   Hence, find the value of n and of k.

6. Given that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{a} + 2\mathbf{b}$ as shown in the figure. AB meets OC at R so that $\overrightarrow{AR} = k\overrightarrow{AB}$ and $\overrightarrow{OR} = n\overrightarrow{OC}$. Express $\overrightarrow{OR}$
   (a) in terms of k, **a** and **b**,
   (b) in terms of n, **a** and **b**.
   Hence, find the value of n and of k.



7. Given that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OP} = \frac{2}{3}\overrightarrow{OA}$ and $\overrightarrow{OQ} = 2\mathbf{b}$, express $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ in terms of **a** and **b**.
   PQ meets AB at R so that $\overrightarrow{PR} = n\overrightarrow{PQ}$ and $\overrightarrow{AR} = k\overrightarrow{AB}$. Express $\overrightarrow{OR}$
   (a) in terms of n, **a** and **b**,
   (b) in terms of k, **a** and **b**.
   Hence find the value of n and of k.

23.3 POSITION VECTORS

In Fig. 23.19 the position of a point P with respect to an origin O is indicated by the directed line segment $\overrightarrow{OP}$. The vector $\overrightarrow{OP}$ is called the **position vector** of P.

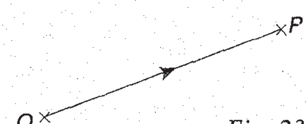


Fig. 23.19

Example 8 Relative to an origin O, the position vectors of the points A and B are **a** and **b**. C is the point such that $OACB$ is a parallelogram and M is the point of intersection of the diagonals OC and AB. Find the position vectors of C and M.

Solution: By the parallelogram law,

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB} = \mathbf{a} + \mathbf{b}$$

The position vector of C is $\underline{\underline{\mathbf{a} + \mathbf{b}}}$.

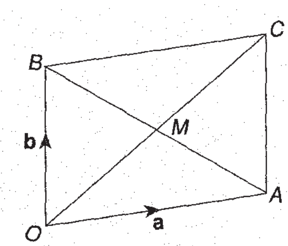


Fig. 23.20

Since the diagonals bisect each other,

$$\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OC} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$$

The position vector of M is $\underline{\underline{\frac{1}{2}(\mathbf{a} + \mathbf{b})}}$.

CHAPTER 23

Exercise 23.1 *(p. 481)*

1. (a) $\overrightarrow{AP}$ (b) $\overrightarrow{AB}$ (c) $\overrightarrow{AD}$ 3. $\frac{1}{2}(\overrightarrow{OB} - \overrightarrow{OA})$, $\frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$

4. (a) $\frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$, $-\frac{3}{2}\overrightarrow{OA} + \overrightarrow{OB}$, $-2\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB}$

(b) $\frac{1}{2}\overrightarrow{OC} + 2\overrightarrow{OA}$, $\overrightarrow{OC} + \frac{3}{2}\overrightarrow{OA}$, $\frac{1}{2}\overrightarrow{OC} - \frac{1}{2}\overrightarrow{OA}$

5. $2\overrightarrow{OB}$ 6. (a) $\mathbf{p} + \mathbf{q}$ (b) $\mathbf{p} - \mathbf{q}$ (c) $\mathbf{q} - \frac{1}{2}\mathbf{p}$; $\sqrt{2}$ units

8. $\mathbf{q} - \sqrt{2}\,\mathbf{p}$ 9. (a) $\overrightarrow{AD}$ (b) $2\overrightarrow{AD}$ (c) $\frac{1}{2}\overrightarrow{AD}$ (d) $5\frac{1}{2}\overrightarrow{AD}$

10. (a) $4\mathbf{q} - 2\mathbf{p}$ (b) $3(\mathbf{q} - \mathbf{p})$

Exercise 23.2 *(p. 485)*

1. $\frac{1}{2}(\mathbf{a} + \mathbf{b})$, $\mathbf{b} - \frac{1}{2}\mathbf{a}$, $\mathbf{b} - \frac{3}{2}\mathbf{a}$ 2. $\mathbf{p} + \mathbf{q}$, $2\mathbf{p} + \mathbf{q}$, $\mathbf{p} + 2\mathbf{q}$, $\mathbf{q} - \mathbf{p}$

3. $s = \frac{4}{7}$, $t = -\frac{5}{7}$ 4. -1

5. (a) $\left(1 - \frac{1}{2}k\right)\mathbf{a} + k\mathbf{c}$ (b) $n\mathbf{c}$; $n = 2$, $k = 2$

6. (a) $(1 - k)\mathbf{a} + k\mathbf{b}$ (b) $n\mathbf{a} + 2n\mathbf{b}$; $n = \frac{1}{3}$, $k = \frac{2}{3}$

7. $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, $\overrightarrow{PQ} = 2\mathbf{b} - \frac{2}{3}\mathbf{a}$

(a) $\frac{2}{3}(1 - n)\mathbf{a} + 2n\mathbf{b}$ (b) $(1 - k)\mathbf{a} + k\mathbf{b}$; $n = \frac{1}{4}$, $k = \frac{1}{2}$

Exercise 23.3 *(p. 490)*

1. $\frac{1}{3}(2\mathbf{a} + \mathbf{b})$, $\frac{1}{3}(\mathbf{a} + 2\mathbf{b})$ 2. $m\mathbf{a} + \mathbf{b}$ 3. $2(\mathbf{a} - \mathbf{b})$, $5(\mathbf{a} - \mathbf{b})$

4. $\overrightarrow{AB} = 2\mathbf{p} - 3\mathbf{q}$, $\overrightarrow{AC} = 5\mathbf{p} + (k - 1)\mathbf{q}$; $k = -\frac{13}{2}$, $\lambda = \frac{2}{5}$

6. $(\lambda - 1)\mathbf{p} + (\lambda + 1)\mathbf{q}$, $(\lambda + 1)\mathbf{q}$, $\lambda = 1$

7. (a) $\frac{1}{2}(\mathbf{p} + \mathbf{q})$ (b) $\mathbf{q} - \mathbf{p}$ (c) $\frac{1}{2}\mathbf{p} - \mathbf{q}$

8. (a) $(1 - n)k\mathbf{a} + \frac{1}{2}n\mathbf{b}$ (b) $\frac{n - 2}{2(n - 1)}$

9. (a) $(m + 1)\mathbf{q} - m\mathbf{p}$ (b) $(3 + n)\mathbf{q} - 9\mathbf{p}$; $m = 9$, $n = 7$, $10\mathbf{q} - 9\mathbf{p}$

10. $\overrightarrow{AQ} = 2\mathbf{b} - \mathbf{a}$, $\overrightarrow{BP} = 3\mathbf{a} - \mathbf{b}$

(a) $(1 - \lambda)\mathbf{a} + 2\lambda\mathbf{b}$ (b) $3\mu\mathbf{a} + (1 - \mu)\mathbf{b}$; $\lambda = \frac{2}{5}$, $\mu = \frac{1}{5}$; $\frac{3}{5}\mathbf{a} + \frac{4}{5}\mathbf{b}$

11. $\frac{2}{5}(1 - l)\mathbf{a} + \frac{2}{5}l\,\mathbf{b}$, $\frac{4}{5}(\mathbf{b} - k\mathbf{a})$; $k = \frac{1}{4}$, $l = \frac{1}{2}$

12. $\overrightarrow{OG} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{d}$, $\overrightarrow{OC} = -\frac{1}{2}\mathbf{a} + \mathbf{b} + \frac{1}{2}\mathbf{d}$, $\overrightarrow{OE} = \mathbf{a} - \mathbf{b} + \mathbf{d}$, $\overrightarrow{OF} = \frac{3}{2}\mathbf{a} - \mathbf{b} + \frac{1}{2}\mathbf{d}$

13. $\overrightarrow{OD} = 3(\mathbf{p} - 2\mathbf{q})$, $\lambda = -4$ 14. $\overrightarrow{OD} = \mathbf{a} + \mathbf{c} - \mathbf{b}$, $\overrightarrow{OM} = \frac{1}{2}(\mathbf{a} + \mathbf{c})$